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Resolvent bounds for pipe Poiseuille flow

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We derive an analytical bound on the resolvent of pipe Poiseuille flow in large parts of the unstable half-plane. We also consider the linearized equations, Fourier transformed in axial and azimuthal directions. For certain combinations of the wavenumbers and the Reynolds number, we derive an analytical bound on the resolvent of the Fourier transformed problem. In particular, this bound is valid for the perturbation which numerical computations indicate to be the perturbation that gives the largest transient growth. Our bound has the same dependence on the Reynolds number as given by the computations.

1. Introduction

Since the pioneer work on pipe flow by Reynolds in the late nineteenth century, hydrodynamical stability theory has experienced great advances. However, some of the most fundamental questions remain unanswered, such as the mechanisms responsible for transition to turbulence. Even in the few simple cases of shear flows where the analytical solutions of the Navier–Stokes equations are available, much is still unknown. It has been shown that plane Couette flow is linearly stable at all Reynolds numbers (Romanov 1973) and that plane Poiseuille flow becomes linearly unstable at $R \approx 5772$ (Orszag 1971). For pipe Poiseuille flow, laminar flow has been observed at $R \approx 10^5$ in highly controlled experiments (Pfenninger 1961) indicating that the flow is linearly stable. Also, numerous numerical computations have been done (see e.g. Lessen, Sadler & Liu 1968; Salwen, Cotton & Grosch 1980; Schmid & Henningson 1994; Trefethen, Trefethen & Schmid 1999) without finding any unstable eigenvalues of the Navier–Stokes equations linearized at the stationary parabolic velocity profile of pipe Poiseuille flow. However, a formal proof of linear stability exists only for axisymmetric disturbances (Herron 1991). Hence, despite the long history of the problem, the question of linear stability of pipe Poiseuille flow remains an open problem.

Even more complicated and unresolved is the question of conditional nonlinear stability of pipe Poiseuille flow. Despite the believed linear stability at all Reynolds numbers, experiments have shown that finite-amplitude perturbations may lead to turbulence at Reynolds numbers larger than the critical Reynolds number $R_c \approx 2000$ (see Draad, Kuiken & Nieuwstadt 1998, and references therein). There is a threshold for the amplitude of the perturbation, below which the flow is stable to all perturbations. This threshold is assumed to behave as $R^{-\beta}$, with $\beta \geqslant 1$, as $R \to \infty$ (Trefethen *et al.* 1993). Determining the correct value of β has proved to be a challenge. Experiments and computations have indicated values in the range $1 \leqslant \beta \leqslant 3/2$ (see e.g. Hof, Juel & Mullin 2003; Meseguer 2003; Shan, Zhang & Nieuwstadt 1998). By careful asymptotic analysis, Chapman (2002) argues that $\beta = 1$ and $\beta = 3/2$ for plane Couette flow and plane Poiseuille flow, respectively. In work not yet published, discussed by e.g. Meseguer & Trefethen (2003), Chapman uses the same technique for pipe Poiseuille flow, with $\beta = 1$ as the resulting asymptotic exponent.

In the last decade, much attention has been devoted to linear transient growth as a possible mechanism for transition to turbulence in shear flows (see e.g. Reddy & Henningson 1993; Trefethen *et al.* 1993, and references therein). This transient growth is due to the non-normality of the operator of the linearized Navier–Stokes equations. More importantly, the operator is increasingly non-normal with increasing Reynolds number. Hence, a small perturbation can exhibit severe short-time growth, owing to linear mechanisms, thus triggering nonlinear effects which lead to turbulence. The transient growth cannot be captured by considering the eigenvalues, since they predict only the exponential decay which eventually follows. More information can be obtained by considering the ε-pseudospectrum or the resolvent.

The ε-pseudospectrum is a generalization of the spectrum. For a linear operator, $\mathscr{L}$, the ε-pseudospectrum is the set of complex numbers, s, such that $\|(sI - \mathscr{L})^{-1}\| \geqslant \varepsilon^{-1}$. Clearly, all eigenvalues are in the ε-pseudospectrum for any value of ε. If the operator is highly non-normal, the ε-pseudospectrum will include large areas around each eigenvalue for small values of ε. This is an indication that the eigenvalues probably give poor information about the short-time behaviour. Also, the ε-pseudospectrum can be used to derive a lower bound on the transient growth (Trefethen *et al.* 1993). Numerical computations of the ε-pseudospectrum for pipe Poiseuille flow have been done by Trefethen *et al.* (1999) and Meseguer & Trefethen (2003).

The term $\mathscr{R}(s) = (sI - \mathscr{L})^{-1}$ in the definition of the ε-pseudospectrum is known as the resolvent of $\mathscr{L}$. Hence, the resolvent is the solution operator of the Laplace transformed initial-value problem $\boldsymbol{u}_t = \mathscr{L}\boldsymbol{u}$. Deriving a bound on the norm of the resolvent in the entire unstable half-plane implies linear stability of the initial-value problem. Also, this bound includes the effects of transient growth and it can also be used for proving conditional nonlinear stability. This was done by Kreiss, Lundbladh & Henningson (1994) who, assuming the bound $\|\mathscr{R}(s)\| \leqslant CR^{\rho}$ in the entire unstable half-plane $\mathrm{Re}(s) \geqslant 0$, proved nonlinear stability of shear flows for perturbations with amplitudes smaller than $\tilde{C}R^{-2\rho-5/4}$. This serves as the only proof of an upper bound on β in the threshold for nonlinear stability of shear flows.

Here, we consider the resolvent of pipe Poiseuille flow. Computations by Meseguer & Trefethen (2003) indicate that the L^2-norm of the resolvent is maximized at $s = 0$ and depends on the Reynolds number as $\|\mathscr{R}(0)\| \sim R^2$. A proof of this bound in the entire unstable half-plane would, besides proving linear stability, make the nonlinear stability result mentioned above directly applicable.

The first result in this paper is a bound on the L^2-norm of the resolvent, obtained from the Laplace transformed and linearized Navier–Stokes equations, in large parts of the unstable half-plane. However, the size of the remaining part grows with R, and the bound is not valid at the point $s = 0$. In order to obtain a bound in the remaining part of the unstable half-plane, we consider the equations in both Cartesian and cylindrical coordinates. When using cylindrical coordinates, there is a well-known reformulation of the equations involving the radial velocity and the radial vorticity. The advantage of this formulation is that the number of unknowns reduces to two. The equations are homogeneous in the axial and azimuthal directions. Hence, Fourier transformation can be used in these directions, with dual variables α and n, respectively. The norm of the resolvent of the original problem can be obtained by maximizing the norm of the resolvent of the Fourier transformed problem with respect to the wavenumbers. Numerical computations indicate that this maximum occurs when $\alpha = 0$ and $n = 1$ (Trefethen *et al.* 1999).

The second result of this paper is an analytical bound on the resolvent for certain combinations of the wavenumbers and the Reynolds number. The bound is valid in the entire unstable half-plane. In particular, the bound is valid for $\alpha = 0$ and $n = 1$, i.e. when the resolvent is believed to be maximized. Our analytical bound has the same R dependence as computations have indicated, i.e. the L^2-norm is proportional to R^2.

This paper is a step towards proving the linear stability of pipe Poiseuille flow. In the case of plane Couette flow, a similar strategy has proved successful. Liefvendahl & Kreiss (2003) derived results for plane Couette flow which are similar to the results in this paper. A resolvent bound for a different combination of the wavenumbers and the Reynolds number has been proved for $s = 0$ by Åsén & Kreiss (2005). In the remaining bounded parameter domain, numerical computations, which could be made rigorous by using interval arithmetic, yield a resolvent bound (Åsén 2005). Together, the computed and the analytical results prove a resolvent bound for $s = 0$.

The paper is organized as follows. In § 2, we state the problem and introduce some notation. A resolvent bound in large parts of the unstable half-plane is derived in § 3. In § 4, we derive bounds for certain combinations of the wavenumbers and the Reynolds number. This is done by considering the equations in both Cartesian and cylindrical coordinates. We also show that the resolvent of the original problem can be obtained by maximizing the resolvent of the Fourier transformed problem with respect to the wavenumbers. We discuss the relation between the norm of the resolvent and transient growth of energy in § 5. In § 6, we discuss what further results are required in order to obtain a bound on the resolvent in the entire unstable half-plane. Finally, we present our conclusions in § 7.

2. The problem

We chose the (Cartesian) coordinate system such that x is the streamwise direction and the pipe radius is one, i.e. the domain is given by

$$\mathscr{D} = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 \leqslant 1\}. \tag{2.1}$$

The (normalized) stationary solution of pipe Poiseuille flow is then given by

$$\boldsymbol{U} = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - y^2 - z^2 \\ 0 \\ 0 \end{pmatrix}. \tag{2.2}$$

In Cartesian coordinates, we use the notation

$$\boldsymbol{u} = (u, v, w)^T = u\boldsymbol{e}_x + v\boldsymbol{e}_y + w\boldsymbol{e}_z$$

for the perturbation, where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are the unit vectors in the x, y and z directions, respectively. Linearizing the Navier–Stokes equations at the stationary solution (2.2) and applying the Laplace transform gives

$$s\boldsymbol{u} + U\boldsymbol{u}_x - \begin{pmatrix} 2yv + 2zw \\ 0 \\ 0 \end{pmatrix} + \nabla p = \frac{1}{R}\Delta\boldsymbol{u} + \boldsymbol{f}, \tag{2.3a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.3b}$$

$$\boldsymbol{u} = 0, \quad (x, y, z) \in \Gamma. \tag{2.3c}$$

Here, $R = U_c a/\nu$ is the Reynolds number, where U_c is the centreline velocity, a the pipe radius and ν the kinematic viscosity, and $\Gamma = \{(x, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1\}$ is the boundary of $\mathscr{D}$.

The resolvent, $\mathscr{R}(s)$, is the solution operator of (2.3*a*–*c*) for a given forcing $\boldsymbol{f}$, i.e. $\mathscr{R}(s) : \boldsymbol{f} \rightarrow \boldsymbol{u}$. We are interested in bounding the L^2-norm of the resolvent in the unstable half-plane, $\mathrm{Re}(s) \geqslant 0$. In particular, we are interested in how the norm of the resolvent depends on the Reynolds number.

We assume that $\boldsymbol{u} \in L^2$ is a smooth solution, i.e. $\boldsymbol{u} \rightarrow 0$ as $|x| \rightarrow \infty$, so that boundary terms vanish when using integration by parts. Without any restriction, we assume $\nabla \cdot \boldsymbol{f} = 0$ and $\boldsymbol{f} \in C_0^\infty$. A non-solenoidal forcing can be divided into a solenoidal part and a part affecting only the pressure (Yudovich 1989). Results for less regular forcing follow from closure arguments. Also, since we are interested in the linear stability for large Reynolds numbers, we consider only $R \geqslant 1$.

We will derive bounds on the resolvent in large parts of the unstable half-plane using the formulation (2.3*a*–*c*) and integration by parts. However, in other parts of the unstable half-plane, this is not possible, at least not in a straightforward way. In those parts, we will derive bounds for some combinations of wavenumbers and the Reynolds number. The geometry of the domain suggests that cylindrical coordinates might be useful, and we will return to this later in the paper.

We use $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ and $\|\boldsymbol{u}\| = \langle \boldsymbol{u}, \boldsymbol{u} \rangle^{1/2}$ to denote the L^2-inner product and L^2-norm, respectively. In Cartesian coordinates, the L^2-inner product is defined as

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \int_{\mathscr{D}} \overline{\boldsymbol{u}} \cdot \boldsymbol{v} \, \mathrm{d}x.$$

As mentioned above, we also use cylindrical coordinates. To avoid confusion, we introduce the corresponding equations and notation later.

3. A resolvent bound in parts of the unstable half-plane

In order to obtain a bound on the resolvent in large parts of the unstable half-plane, we consider the linearized Navier–Stokes equations in Cartesian coordinates, (2.3*a*–*c*). First, we define the following parts of the complex plane (see figure 1).

$$\Sigma = \left\{ s \in \mathbb{C} : \mathrm{Re}(s) - 7 + \frac{1}{2R} |\mathrm{Im}(s)| \geqslant 0 \right\}, \qquad (3.1a)$$

$$\Sigma^- = \{ s \notin \Sigma : \mathrm{Re}(s) \geqslant 0 \}. \qquad (3.1b)$$

We can derive a bound on the resolvent using only integration by parts. The result is summarized in the following theorem.

THEOREM 3.1. *If $s \in \Sigma$, where Σ is defined by (3.1a), then the resolvent is bounded by*

$$\|\mathscr{R}(s)\| \leqslant CR,$$

where C is a constant independent of R. Also, for $s \in \Sigma$ we have $\|\mathscr{R}(s)\| \rightarrow 0$ as $|s| \rightarrow \infty$.

Proof. Scalar multiply (2.3*a*) with $\boldsymbol{u}$. For the term involving the pressure, we have by using integration by parts, (2.3*b*) and (2.3*c*) so that

$$\langle \boldsymbol{u}, \nabla p \rangle = -\langle \nabla \cdot \boldsymbol{u}, p \rangle = 0.$$

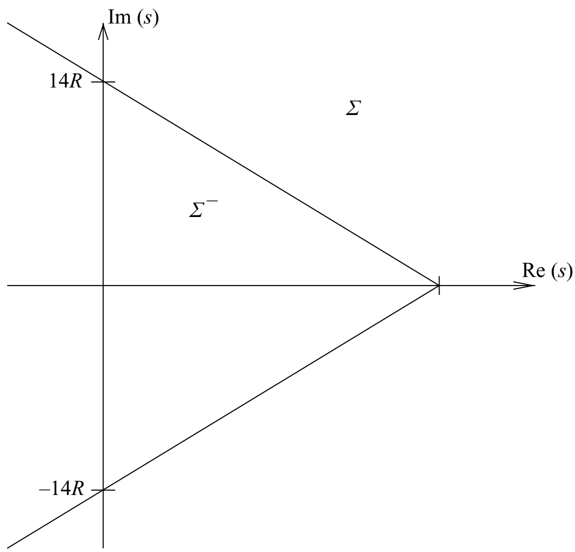

FIGURE 1. The L^2-norm of the resolvent of pipe Poiseuille flow is bounded by $\|\mathscr{R}(s)\| \leqslant CR$ when $s \in \Sigma$. Also, the resolvent tends to zero as $|s| \to \infty$ when $s \in \Sigma$.

Using the triangle inequality and $a + b \leqslant \sqrt{2}\sqrt{a^2 + b^2}$ yields

$$2|\langle u, yv + zw\rangle| \leqslant 2\|\boldsymbol{u}\|\ \|yv + zw\| \leqslant 2\|\boldsymbol{u}\|(\|v\| + \|w\|) \leqslant 2\|\boldsymbol{u}\|\sqrt{2}\|\boldsymbol{u}\| < 3\|\boldsymbol{u}\|^2$$

and, since U is real and independent of x, we also have

$$\langle \boldsymbol{u}, U\boldsymbol{u}_x\rangle = -\langle \boldsymbol{u}_x, U\boldsymbol{u}\rangle = -\overline{\langle \boldsymbol{u}, U\boldsymbol{u}_x\rangle} \Rightarrow \langle \boldsymbol{u}, U\boldsymbol{u}_x\rangle \in \mathrm{Im}.$$

Hence, using integration by parts and taking the real part gives

$$(\mathrm{Re}(s) - 3)\|\boldsymbol{u}\|^2 + \frac{1}{R}(\|\boldsymbol{u}_x\|^2 + \|\boldsymbol{u}_y\|^2 + \|\boldsymbol{u}_z\|^2) \leqslant \|\boldsymbol{u}\|\ \|\boldsymbol{f}\|. \tag{3.2}$$

Similarly, using integration by parts, taking the imaginary part and using $|\langle \boldsymbol{u}, U\boldsymbol{u}_x\rangle| \leqslant \|\boldsymbol{u}\|^2/4 + \|\boldsymbol{u}_x\|^2$ gives

$$\left(|\mathrm{Im}(s)| - \tfrac{1}{4} - 3\right)\|\boldsymbol{u}\|^2 - \|\boldsymbol{u}_x\|^2 \leqslant \|\boldsymbol{u}\|\ \|\boldsymbol{f}\|. \tag{3.3}$$

Dividing (3.3) by R, adding to (3.2) and dividing both sides by $\|\boldsymbol{u}\|$ yields

$$\left(\mathrm{Re}(s) - 3 + \frac{1}{R}|\mathrm{Im}(s)| - \frac{13}{4R}\right)\|\boldsymbol{u}\| \leqslant \left(1 + \frac{1}{R}\right)\|\boldsymbol{f}\|.$$

Using $R \geqslant 1$, it follows that

$$\left(\mathrm{Re}(s) + \frac{1}{R}|\mathrm{Im}(s)| - \tfrac{25}{4}\right)\|\boldsymbol{u}\| \leqslant 2\|\boldsymbol{f}\|.$$

Hence, if $s \in \Sigma$, we have

$$\|\boldsymbol{u}\| \leqslant \frac{4R}{|\mathrm{Im}(s)|}\|\boldsymbol{f}\|. \tag{3.4}$$

We also have from (3.2) that

$$\|\boldsymbol{u}\| \leqslant \frac{1}{\mathrm{Re}(s)-3}\|\boldsymbol{f}\|. \tag{3.5}$$

Using either (3.4) or (3.5), depending on s, the result follows. □

Remark. As seen from (3.5), it is enough that $\mathrm{Re}(s) > 3$ (actually $\mathrm{Re}(s) > 2\sqrt{2}$) for the resolvent to be bounded. Hence, we could make the part of the unstable half-plane where theorem 3.1 does not hold, i.e. Σ^- defined by (3.1*b*), somewhat smaller if desired. However, since Σ^- grows as R increases, this is of minor interest. Also, the estimate (3.5) gives an R-independent bound on the resolvent when $\mathrm{Re}(s) > 3$.

In order to prove linear stability and also nonlinear stability for sufficiently small perturbations, we wish to bound the resolvent in the entire unstable half-plane. The rest of the paper is concerned with how a bound on the resolvent could also be derived in the part of the unstable half-plane not covered by theorem 3.1.

4. Resolvent bounds for certain wavenumbers

Here, we consider the Fourier transformed linearized Navier–Stokes equations. We derive resolvent bounds for certain combinations of wavenumbers in relation to the Reynolds number.

First, we bound the resolvent when the wavenumber in the axial direction, α, is sufficiently large compared to the Reynolds number. For this, we use the linearized Navier–Stokes equations in Cartesian coordinates and Fourier transformed in the axial direction.

Next, we derive a resolvent bound when the product of the axial wavenumber, α, and the Reynolds number, R, is sufficiently small. In this case, we use the linearized Navier–Stokes equations in cylindrical coordinates.

Finally, we bound the resolvent when the azimuthal wavenumber, n, is sufficiently large compared to the product of the axial wavenumber, α, and the Reynolds number. In this case, we use a well-known formulation involving the radial velocity and the radial vorticity.

At the end of this section, we discuss the relation between the norm of the resolvent to the original problem, (2.3*a*–*c*), and the norm of the resolvent to the Fourier transformed problems.

4.1. *Cartesian coordinates*

Since the coefficients in (2.3*a*–*c*) are independent of x, we may apply the Fourier transform, yielding

$$s\hat{\boldsymbol{u}} + \mathrm{i}\alpha U\hat{\boldsymbol{u}} - \begin{pmatrix} 2y\hat{v} + 2z\hat{w} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{i}\alpha\hat{p} \\ \hat{p}_y \\ \hat{p}_z \end{pmatrix} = \frac{1}{R}\hat{\Delta}\hat{\boldsymbol{u}} + \hat{\boldsymbol{f}}, \tag{4.1a}$$

$$\mathrm{i}\alpha\hat{u} + \hat{v}_y + \hat{w}_z = 0, \tag{4.1b}$$

$$\hat{\boldsymbol{u}} = 0, \quad (y,z) \in \Gamma. \tag{4.1c}$$

Here, $\hat{\Delta} = (\partial_y^2 + \partial_z^2 - \alpha^2)$ and the domain is $\mathscr{D} = \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 \leqslant 1\}$ with boundary $\Gamma = \{(y,z) \in \mathbb{R}^2 : y^2 + z^2 = 1\}$.

We define, in analogy with the original problem, $\hat{\mathscr{R}}(s,\alpha)$ to be the solution operator of (4.1*a*–*c*) where $\alpha \in \mathbb{R}$ is to be considered as another parameter, i.e. $\hat{\mathscr{R}}(s,\alpha) : \hat{\boldsymbol{f}} \to \hat{\boldsymbol{u}}$. Note that $\|\cdot\|$ now denotes the L^2-norm over the two-dimensional unit disk.

Using integration by parts, we obtain the following lemma.

LEMMA 4.1. *For all* α *and* R *such that*

$$\alpha^2 \geqslant 4R,$$

the bound

$$\|\hat{\mathscr{R}}(s,\alpha)\| \leqslant 1 \tag{4.2}$$

holds in the entire unstable half-plane $Re(s) \geqslant 0$. *Also,* $\|\hat{\mathscr{R}}(s,\alpha)\| \to 0$ *as* $|\alpha| \to \infty$.

Proof. As in the proof of theorem 3.1, scalar multiplying (4.1*a*) with $\hat{\boldsymbol{u}}$, using integration by parts, (4.1*b*), (4.1*c*) and taking the real part gives

$$(\text{Re}(s) - 3)R\|\hat{\boldsymbol{u}}\|^2 + \|\hat{\boldsymbol{u}}_y\|^2 + \|\hat{\boldsymbol{u}}_z\|^2 + \alpha^2\|\hat{\boldsymbol{u}}\|^2 \leqslant R\|\hat{\boldsymbol{u}}\|\,\|\hat{\boldsymbol{f}}\| \leqslant \frac{R}{2}\|\hat{\boldsymbol{u}}\|^2 + \frac{R}{2}\|\hat{\boldsymbol{f}}\|^2.$$

Rearranging the terms and using $\text{Re}(s) \geqslant 0$ yields

$$\|\hat{\boldsymbol{u}}\|^2 \leqslant \left(\alpha^2 - \frac{7R}{2}\right)^{-1} \frac{R}{2}\|\hat{\boldsymbol{f}}\|^2 \tag{4.3}$$

and the lemma easily follows. □

4.2. *Cylindrical coordinates*

Here, we derive bounds when the product of the wavenumber in the axial direction, α, and the Reynolds number, R, is sufficiently small and when the wavenumber in the azimuthal direction, n, is sufficiently large.

We use the notation

$$\boldsymbol{u} = (u, v, w) = u\boldsymbol{e}_x + v\boldsymbol{e}_r + w\boldsymbol{e}_\theta$$

for the perturbation, where $\boldsymbol{e}_x$, $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are the unit vectors in the x, r and θ directions, respectively. The stationary solution is now given by

$$\boldsymbol{U} = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - r^2 \\ 0 \\ 0 \end{pmatrix}.$$

In cylindrical coordinates, the coefficients in the linearized and Laplace transformed Navier–Stokes equations depend only on r. Hence, we may apply the Fourier transform in the x and θ directions, with dual variables α and n, respectively. The resulting equations are

$$s\tilde{u} + \mathrm{i}\alpha U\tilde{u} - 2r\tilde{v} + \mathrm{i}\alpha\tilde{p} = R^{-1}\tilde{\Delta}\tilde{u} + \tilde{f}^x, \tag{4.4a}$$

$$s\tilde{v} + \mathrm{i}\alpha U\tilde{v} + \tilde{p}' = R^{-1}(\tilde{\Delta}\tilde{v} - r^{-2}\tilde{v} - 2\mathrm{i}nr^{-2}\tilde{w}) + \tilde{f}^r, \tag{4.4b}$$

$$s\tilde{w} + \mathrm{i}\alpha U\tilde{w} + r^{-1}\mathrm{i}n\tilde{p} = R^{-1}(\tilde{\Delta}\tilde{w} + 2\mathrm{i}nr^{-2}\tilde{v} - r^{-2}\tilde{w}) + \tilde{f}^\theta, \tag{4.4c}$$

$$\mathrm{i}\alpha\tilde{u} + r^{-1}(r\tilde{v})' + \mathrm{i}nr^{-1}\tilde{w} = 0, \tag{4.4d}$$

$$\tilde{\boldsymbol{u}} = 0, \quad r = 1, \tag{4.4e}$$

where the prime denotes differentiation with respect to r. Here, the forcing is $\tilde{\boldsymbol{f}} = (\tilde{f}^x, \tilde{f}^r, \tilde{f}^\theta)$, the domain is $\mathscr{D} = \{r \in \mathbb{R} : r \in [0, 1]\}$ and the Laplacian is given by

$$\tilde{\Delta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \alpha^2 - \frac{n^2}{r^2}.$$

We define $\tilde{\mathscr{R}}(s, \alpha, n)$ to be the solution operator of (4.4*a*–*e*) with α and n as parameters, i.e. $\tilde{\mathscr{R}}(s, \alpha, n) : \tilde{\boldsymbol{f}} \to \tilde{\boldsymbol{u}}$. Clearly, $\boldsymbol{u}$ is periodic in the azimuthal direction. Hence, n only takes integer values, i.e. $n \in \mathbb{Z}$, and, as before, $\alpha \in \mathbb{R}$.

The only remaining space dimension is r, and since we use cylindrical coordinates, the scalar product is given by

$$\langle \tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}} \rangle = \int_0^1 \overline{\tilde{\boldsymbol{u}}} \cdot \tilde{\boldsymbol{v}} r \, \mathrm{d}r. \tag{4.5}$$

In this section, $\| \cdot \|$ denotes the norm induced by (4.5), i.e. over the one-dimensional domain $r \in [0, 1]$. Using Parseval's formula, this norm can be related to the L^2-norm over the unit disk and to the L^2-norm over the entire three-dimensional domain (2.1). This is discussed in §4.3.

We first consider axisymmetric perturbations, i.e. $n = 0$, in which case we derive the following lemma.

LEMMA 4.2. *When $n = 0$ and $|\alpha R| \leqslant 1/16$, the bound*

$$\|\tilde{\mathscr{R}}(s, \alpha, 0)\| \leqslant CR$$

holds in the entire unstable half-plane, $Re(s) \geqslant 0$. Here, C is a constant independent of α and R.

Proof. First, with the scalar product (4.5) we have by using (4.4*d*), (4.4*e*) and $n = 0$ that

$$\langle \tilde{u}, \mathrm{i}\alpha \tilde{p} \rangle + \langle \tilde{v}, \tilde{p}' \rangle = -\langle \mathrm{i}\alpha \tilde{u} + r^{-1}(r\tilde{v})', \tilde{p} \rangle = 0,$$

$$\mathrm{i}\alpha(\langle \tilde{u}, U\tilde{u} \rangle + \langle \tilde{v}, U\tilde{v} \rangle + \langle \tilde{w}, U\tilde{w} \rangle) \in \mathrm{Im}.$$

Note that the boundary term from the integration by parts vanishes by using (4.4*e*) at $r = 1$ and by using that $\tilde{v}$, $\tilde{p}$ are bounded at $r = 0$, i.e. $\tilde{v}\tilde{p}r|_{r=0} = 0$. In the rest of this proof, we use (4.4*e*) and the fact that $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}'$ are bounded at $r = 0$ in order to remove boundary terms appearing when using integration by parts.

Now, scalar multiplying (4.4*a*) with $\tilde{u}$, (4.4*b*) with $\tilde{v}$ and (4.4*c*) with $\tilde{w}$, using integration by parts, taking the real part and adding the resulting equations yields

$$\begin{aligned}(\mathrm{Re}(Rs) + \alpha^2)\|\tilde{\boldsymbol{u}}\|^2 &+ \|\tilde{\boldsymbol{u}}'\|^2 + \|r^{-1}\tilde{v}\|^2 + \|r^{-1}\tilde{w}\|^2 \\ &\leqslant R\|\tilde{u}\|\|\tilde{f}^x\| + R\|\tilde{v}\|\|\tilde{f}^r\| + R\|\tilde{w}\|\|\tilde{f}^\theta\| + |2R\langle \tilde{u}, r\tilde{v} \rangle| \\ &\leqslant \tfrac{1}{4}\|\tilde{\boldsymbol{u}}\|^2 + 2R^2\|\tilde{\boldsymbol{f}}\|^2 + |2R\langle \tilde{u}, r\tilde{v} \rangle|. \end{aligned} \tag{4.6}$$

For the last term on the right-hand side, we required the inequality

$$|r\tilde{u}|_\infty \leqslant 2\|\tilde{u}\| + \|\tilde{u}'\|, \tag{4.7}$$

where $| \cdot |_\infty$ denotes the L^∞—norm. In order to prove (4.7), note that since $\tilde{u}$ is continuous, there exist r_m and r_M such that

$$|r_m\tilde{u}(r_m)| = \min_{r\in[0,1]} |r\tilde{u}(r)| \leqslant \|r\tilde{u}\| \leqslant \|\tilde{u}\|, \quad |r_M\tilde{u}(r_M)| = \max_{r\in[0,1]} |r\tilde{u}(r)| = |r\tilde{u}|_\infty.$$

Now, (4.7) follows from

$$\begin{aligned}|r_M\tilde{u}(r_M)|^2 - |r_m\tilde{u}(r_m)|^2 &= \int_{r_m}^{r_M} (r^2\tilde{u}^2)' \, \mathrm{d}r = \int_{r_m}^{r_M} 2r\tilde{u}^2 + 2r^2\tilde{u}\tilde{u}' \, \mathrm{d}r \\ &\leqslant 2\|\tilde{u}\|^2 + 2\|\tilde{u}\|\|\tilde{u}'\| \leqslant 3\|\tilde{u}\|^2 + \|\tilde{u}'\|^2\end{aligned}$$

and $(4\|\tilde{u}\|^2 + \|\tilde{u}'\|^2)^{1/2} \leqslant 2\|\tilde{u}\| + \|\tilde{u}'\|$.

With $n=0$ we have, from (4.4*d*), $(r\tilde{v})' = -\mathrm{i}\alpha\tilde{u}r$ or $r\tilde{v} = -\mathrm{i}\alpha\int_0^r \tilde{u}s\,\mathrm{d}s$. Using this and (4.7) yields

$$\begin{aligned}|2R\langle\tilde{u}, r\tilde{v}\rangle| &= 2R\left|\int_0^1 \bar{\tilde{u}}\left(\mathrm{i}\alpha\int_0^r \tilde{u}s\,\mathrm{d}s\right) r\,\mathrm{d}r\right| \leqslant 2|\alpha R||r\tilde{u}|_\infty \int_0^1 |\bar{\tilde{u}}|r^2\,\mathrm{d}r \\ &\leqslant 2|\alpha R|(2\|\tilde{u}\| + \|\tilde{u}'\|)\left(\int_0^1 |\bar{\tilde{u}}|^2 r\,\mathrm{d}r\right)^{1/2}\left(\int_0^1 r^3\,\mathrm{d}r\right)^{1/2} \\ &= 2|\alpha R|(2\|\tilde{u}\| + \|\tilde{u}'\|)\|\tilde{u}\|\tfrac{1}{2} \leqslant |\alpha R|\left(3\|\tilde{u}\|^2 + \tfrac{1}{4}\|\tilde{u}'\|^2\right).\end{aligned} \tag{4.8}$$

We also need the Poincaré type inequality

$$\|\tilde{\boldsymbol{u}}\|^2 \leqslant \tfrac{1}{4}\|\tilde{\boldsymbol{u}}'\|^2. \tag{4.9}$$

Consider one component of $\tilde{\boldsymbol{u}}$, e.g. $\tilde{u}$. Using $\tilde{u} = -\int_r^1 \tilde{u}'\mathrm{d}s$ yields

$$\begin{aligned}\|\tilde{u}\|^2 &= \int_0^1\left[\int_r^1 \tilde{u}'\,\mathrm{d}s\right]^2 r\,\mathrm{d}r \leqslant \int_0^1\left[\left(\int_r^1 |\tilde{u}'|^2 s\,\mathrm{d}s\right)^{1/2}\left(\int_r^1 s^{-1}\,\mathrm{d}s\right)^{1/2}\right]^2 r\,\mathrm{d}r \\ &\leqslant \|\tilde{u}'\|^2\int_0^1 |\ln(r)|r\,\mathrm{d}r = \tfrac{1}{4}\|\tilde{u}'\|^2.\end{aligned}$$

Doing the same for $\tilde{v}$ and $\tilde{w}$ gives (4.9).

From (4.6), (4.8) and (4.9) we have, using $\mathrm{Re}(Rs) \geqslant 0$,

$$\left(\alpha^2 + \tfrac{3}{4} - 3|\alpha R|\right)\|\tilde{\boldsymbol{u}}\|^2 + \left(\tfrac{3}{4} - \tfrac{1}{4}|\alpha R|\right)\|\tilde{\boldsymbol{u}}'\|^2 \leqslant 2R^2\|\tilde{\boldsymbol{f}}\|^2.$$

The condition $|\alpha R| \leqslant 1/16$ is more than enough to ensure that the term in parentheses on the left-hand side is positive, and the lemma follows. □

In order to obtain a resolvent bound when the azimuthal wavenumber, n, is sufficiently large, we consider a well-known reformulation of the problem. This formulation is obtained by eliminating the pressure and formulating equations for the radial velocity, $\tilde{v}$, and the radial vorticity, $\tilde{\eta}$ (Burridge & Drazin 1969). The resulting equations are

$$\frac{1}{R}\mathrm{T}^2\Phi - (\mathrm{i}\alpha U + s)\mathrm{T}\Phi + k^2\mathrm{i}\alpha r\left(\frac{U'}{k^2 r}\right)'\Phi - \frac{2\alpha n}{R}\mathrm{T}\Omega = -\mathrm{T}\xi, \tag{4.10a}$$

$$\frac{1}{R}\mathrm{S}\Omega - (\mathrm{i}\alpha U + s)\Omega + \frac{2\alpha n}{Rk^4 r^4}\mathrm{T}\Phi + \frac{\mathrm{i}nU'}{k^2 r^3}\Phi = -\chi. \tag{4.10b}$$

Here, the prime denotes differentiation with respect to r, $k^2 = \alpha^2 + n^2/r^2$ and

$$\Phi = -\mathrm{i}r\tilde{v}, \qquad \Omega = \frac{\alpha r\tilde{w} - n\tilde{u}}{k^2 r^2} = \frac{-\tilde{\eta}}{\mathrm{i}k^2 r},$$

$$\xi = -\mathrm{i}r\tilde{f}^r, \qquad \chi = \frac{\alpha r\tilde{f}^\theta - n\tilde{f}^x}{k^2 r^2},$$

$$\mathrm{T} = k^2 r\frac{\partial}{\partial r}\left(\frac{1}{k^2 r}\frac{\partial}{\partial r}\right) - k^2, \qquad \mathrm{S} = \frac{1}{k^2 r^3}\frac{\partial}{\partial r}\left(k^2 r^3\frac{\partial}{\partial r}\right) - k^2. \tag{4.11a,b}$$

The corresponding boundary conditions are given by (see e.g. Schmid & Henningson 2001),

$$\left.\begin{array}{ll} r=1: & \Phi=\Phi'=\Omega=0, \\ r=0,\ n=0: & \Phi=\Phi'=0, \\ r=0,\ |n|=1: & \Phi=\Omega=0, \Phi' \text{ finite}, \\ r=0,\ |n|\geqslant 2: & \Phi=\Phi'=\Omega=0. \end{array}\right\} \tag{4.12}$$

The two variables Φ and Ω completely describe the system. The original variables can be recovered from

$$\tilde{u}=-\frac{\alpha}{k^2 r}\Phi'-n\Omega, \quad \tilde{v}=\frac{\mathrm{i}\Phi}{r}, \quad \tilde{w}=-\frac{n}{k^2 r^2}\Phi'+\alpha r\Omega.$$

Hence, the L^2-norm can be computed as

$$\|\tilde{\boldsymbol{u}}\|^2=\|r^{-1}\Phi\|^2+\|k^{-1}r^{-1}\Phi'\|^2+\|kr\Omega\|^2, \tag{4.13}$$

and similarly, $\|\tilde{\boldsymbol{f}}\|^2$ can be computed as $\|\tilde{\boldsymbol{f}}\|^2=\|r^{-1}\xi\|^2+\|k^{-1}r^{-1}\xi'\|^2+\|kr\chi\|^2$.

Using (4.10*a, b*), we obtain the following lemma.

LEMMA 4.3. *For all* α, n *and* R *such that*

$$n^2 \geqslant 16|\alpha R|,$$

the bound

$$\|\tilde{\mathscr{R}}(s,\alpha,n)\| \leqslant CR^2$$

holds in the entire unstable half-plane, $\mathrm{Re}(s)\geqslant 0$. *Here,* C *is a constant independent of* α, n *and* R. *Also, for any fixed* α *and* R, $\|\tilde{\mathscr{R}}(s,\alpha,n)\| \to 0$ *as* $|n|\to\infty$.

Proof. The rather lengthy proof is given in the Appendix. □

4.3. *The relation between the original problem and the Fourier transformed problems*

Here, we discuss how the bounds on $\|\hat{\mathscr{R}}(s,\alpha)\|$ and $\|\tilde{\mathscr{R}}(s,\alpha,n)\|$ can be used to derive a bound on $\|\mathscr{R}(s)\|$. The arguments closely follow those used by Liefvendahl & Kreiss (2003). We start by proving the following theorem.

THEOREM 4.2. *For all* α, n *and* R *such that at least one of the inequalities*

$$|\alpha R| \leqslant \tfrac{1}{16}, \quad |\alpha|^3 \geqslant 4|\alpha R|, \quad n^2 \geqslant 16|\alpha R| \tag{4.14a–c}$$

hold, there is a constant C, *independent of* α, n, R *and* $Re(s)\geqslant 0$, *such that*

$$\|\tilde{\mathscr{R}}(s,\alpha,n)\| \leqslant CR^2. \tag{4.15}$$

Also, $\|\tilde{\mathscr{R}}(s,\alpha,n)\| \to 0$ *as* $|\alpha|+|n|\to\infty$.

Proof. Lemma 4.3 gives (4.15) when (4.14*c*) holds. Also, using lemma 4.3 when $n\neq 0$ and lemma 4.2 when $n=0$ yields (4.15) when (4.14*a*) holds. Proving that lemma 4.1 is also valid for $\tilde{\mathscr{R}}(s,\alpha,n)$ for any value of n will give (4.15) when (4.14*b*) holds, and will thus be the equivalent of proving the theorem.

When we derived lemma 4.1, we used the linearized Navier–Stokes equations in Cartesian coordinates, Fourier transformed in the axial direction. Clearly, the L^2-norm over the unit disk of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{f}}$ is the same when using cylindrical coordinates as when using Cartesian coordinates. Hence, we may assume that $\hat{\mathscr{R}}(s,\alpha)$ in lemma 4.1 is the solution operator of the once Fourier transformed problem given in cylindrical coordinates.

In order to prove that lemma 4.1 also holds for $\tilde{\mathscr{R}}(s,\alpha,n)$, we require Parseval's formula, given in this case by

$$\|\hat{\boldsymbol{u}}\|^2 = \int_0^{2\pi}\int_0^1 |\hat{\boldsymbol{u}}(r,\alpha,\theta)|^2 r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\sum_{n=-\infty}^{\infty}\int_0^1 |\tilde{\boldsymbol{u}}(r,\alpha,n)|^2 r\,\mathrm{d}r = 2\pi\sum_{n=-\infty}^{\infty}\|\tilde{\boldsymbol{u}}\|^2.$$

Assume that lemma 4.1 does not hold for $\tilde{\mathscr{R}}(s,\alpha,n)$ for all n. This means that there exists an α^* with $|\alpha^*| > 2\sqrt{R}$, an n^* and a forcing $\tilde{\boldsymbol{f}}^*(r)$ with $\|\tilde{\boldsymbol{f}}^*\| = 1$ such that

$$\|\tilde{\mathscr{R}}(s,\alpha^*,n^*)\tilde{\boldsymbol{f}}^*\| > 1. \tag{4.16}$$

Denote by $\tilde{\boldsymbol{u}}^*(r)$ the corresponding solution. Now, consider the inverse transform of $\tilde{\boldsymbol{f}}^*(r)$, i.e. the forcing

$$\hat{\boldsymbol{f}}^*(r,\theta) = \tilde{\boldsymbol{f}}^*(r)\exp(\mathrm{i}n^*\theta)$$

with corresponding solution $\hat{\boldsymbol{u}}^*(r,\theta)$. Using Parseval's formula and (4.16), we then have

$$\|\hat{\boldsymbol{u}}^*\|^2 = 2\pi\sum_{n=-\infty}^{\infty}\|\tilde{\boldsymbol{u}}^*\|^2 = 2\pi\|\tilde{\mathscr{R}}(s,\alpha^*,n^*)\tilde{\boldsymbol{f}}^*\|^2 > 2\pi.$$

Since $\|\hat{\boldsymbol{f}}^*\| = 2\pi$, this would imply $\|\hat{\mathscr{R}}(s,\alpha^*)\| > 1$, i.e. that lemma 4.1 does not hold for $\hat{\mathscr{R}}(s,\alpha)$ either. Hence, the bound (4.2) of lemma 4.1 holds also for $\tilde{\mathscr{R}}(s,\alpha,n)$ for all values of n.

The proof that $\|\tilde{\mathscr{R}}(s,\alpha,n)\| \to 0$ as $|\alpha| \to \infty$ is almost identical. From (4.3), we have $\|\hat{\mathscr{R}}(s,\alpha)\| \leqslant (2\alpha^2 - 7R)^{-1}R$. By the same arguments as above, this bound also holds for $\|\tilde{\mathscr{R}}(s,\alpha,n)\|$ for all n. Hence, it follows that $\|\tilde{\mathscr{R}}(s,\alpha,n)\| \to 0$ as $|\alpha| \to \infty$ for all values of n. This proves that lemma 4.1 holds also for $\tilde{\mathscr{R}}(s,\alpha,n)$ for all values of n and theorem 4.2 is thus proved. □

Next, we show that $\tilde{\mathscr{R}}(s,\alpha,n)$ is related to $\mathscr{R}(s)$ by the following relation

$$\|\mathscr{R}(s)\| = \max_{\alpha,n}\|\tilde{\mathscr{R}}(s,\alpha,n)\|. \tag{4.17}$$

Note that $\|\cdot\|$ on the left-hand side denotes the L^2-norm over the entire three-dimensional domain (2.1) and on the right-hand side denotes the norm induced by the scalar product (4.5).

The proof of (4.17) is straightforward. Note that in the proof of theorem 4.2, we actually proved

$$\|\hat{\mathscr{R}}(s,\alpha)\| \geqslant \max_{n}\|\tilde{\mathscr{R}}(s,\alpha,n)\|. \tag{4.18}$$

Here, we have used max instead of sup since $\|\tilde{\mathscr{R}}(s,\alpha,n)\| \to 0$ as $|n| \to \infty$ by lemma 4.3, i.e. $\|\tilde{\mathscr{R}}(s,\alpha,n)\|$ attains a maximal value with respect to n. The opposite inequality of (4.18) follows from

$$\|\tilde{\boldsymbol{u}}\|^2 = \|\tilde{\mathscr{R}}(s,\alpha,n)\tilde{\boldsymbol{f}}\|^2 \leqslant \left(\max_{n}\|\tilde{\mathscr{R}}(s,\alpha,n)\|\right)^2\|\tilde{\boldsymbol{f}}\|^2.$$

Using this with Parseval's formula gives $\|\hat{\mathscr{R}}(s,\alpha)\| \leqslant \max_n\|\tilde{\mathscr{R}}(s,\alpha,n)\|$, and we thus have

$$\|\hat{\mathscr{R}}(s,\alpha)\| = \max_{n}\|\tilde{\mathscr{R}}(s,\alpha,n)\|.$$

In order to prove (4.17), we must now prove

$$\|\mathscr{R}(s)\| = \max_{\alpha} \|\hat{\mathscr{R}}(s,\alpha)\|. \tag{4.19}$$

For this, we require $\|\hat{\mathscr{R}}(s,\alpha)\| \to 0$ as $|\alpha| \to \infty$, in order to ensure that $\|\hat{\mathscr{R}}(s,\alpha)\|$ attains a maximal value with respect to α; but this follows from lemma 4.1. Now, the rest of the proof is similar to the proof above, although some care must be taken because $\alpha \in \mathbb{R}$, i.e. α does not only take integer values as n does. The proof of (4.19) was done by Liefvendahl & Kreiss (2003) and (4.17) follows.

5. Relation between the resolvent and transient growth

For pipe Poiseuille flow, numerical computations concerning the energy of an initial perturbation as a function of time have been done by e.g. Schmid & Henningson (1994) and Meseguer & Trefethen (2003). The results show that a substantial initial growth of energy is possible despite the stable eigenvalues. In this section, we relate the norm of the resolvent to this transient growth of energy.

Consider the initial-value problem

$$\boldsymbol{u}_t = \mathscr{L}\boldsymbol{u}, \tag{5.1a}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{5.1b}$$

where $\mathscr{L}$ is a linear operator independent of time. If we denote the solution operator of (5.1*a*–*b*) by $e^{t\mathscr{L}}$, we have

$$\boldsymbol{u}(t) = e^{t\mathscr{L}}\boldsymbol{u}_0. \tag{5.2}$$

Since $\|u(t)\| \leqslant \|e^{t\mathscr{L}}\|\ \|\boldsymbol{u}_0\|$, we may use $\|e^{t\mathscr{L}}\|$ as a measure of the largest possible growth (in the norm used) as a function of time. In hydrodynamic stability, the L^2-norm is typically used, since the square of the L^2-norm can be interpreted as an energy.

Assume that the spectrum of $\mathscr{L}$ is in the left half-plane, i.e. $\lim_{t\to\infty} \|\boldsymbol{u}(t)\| = 0$. Clearly, this implies $\lim_{t\to\infty} \|e^{t\mathscr{L}}\| = 0$. Now, if $\mathscr{L}$ is a normal operator, we have $\|e^{t\mathscr{L}}\| \leqslant 1$, $\forall t \geqslant 0$. This means that no growth of the norm of the solution is possible for any initial data, $\boldsymbol{u}_0$. However, if $\mathscr{L}$ is non-normal, the norm of the solution can experience an initial growth, i.e. $\|e^{t\mathscr{L}}\| > 1$ for some times $t > 0$, before eventually decaying.

In order to derive a relation between the resolvent and the solution operator, $e^{t\mathscr{L}}$, we apply the Laplace transform to (5.1*a*). The solution can then be written as

$$\tilde{\boldsymbol{u}}(s) = (s\boldsymbol{I} - \mathscr{L})^{-1}\boldsymbol{u}_0 \equiv \mathscr{R}(s)\boldsymbol{u}_0, \tag{5.3}$$

where $\boldsymbol{I}$ is the identity operator and $\mathscr{R}(s) = (s\boldsymbol{I} - \mathscr{L})^{-1}$ is the resolvent of $\mathscr{L}$.

Applying the Laplace transform to (5.2) and comparing with (5.3), we find from the definition of the Laplace transform that

$$\mathscr{R}(s) = \int_0^\infty e^{-st} e^{t\mathscr{L}}\, dt. \tag{5.4}$$

If $\|e^{t\mathscr{L}}\|$ is large, one also expects the norm of the resolvent to be large. Note that the resolvent integrates the effects of transient growth over time. Thus, the norm of the resolvent can be significantly larger than $\|e^{t\mathscr{L}}\|$.

The effects of transient growth can be illustrated by a simple example taken from Schmid & Henningson (1994), and we refer to this paper for further details. Consider the 2×2 model problem (5.1) for $\boldsymbol{u} = (u, v)^T$ and $\mathscr{L}$ given by

$$\mathscr{L} = \begin{pmatrix} -1/R & 0 \\ 1 & -2/R \end{pmatrix}. \tag{5.5}$$

Clearly, the eigenvalues of $\mathscr{L}$ are in the left-hand half-plane for all $R > 0$. The solution operator of this model problem is

$$\mathrm{e}^{t\mathscr{L}} = \begin{pmatrix} \mathrm{e}^{-t/R} & 0 \\ -\left(\mathrm{e}^{-2t/R} - \mathrm{e}^{-t/R}\right)R & \mathrm{e}^{-2t/R} \end{pmatrix}.$$

From this, we see that $\sup_{t \geqslant 0} \|\mathrm{e}^{t\mathscr{L}}\| \sim R$ and that the maximum is attained at a time $t \sim R$. Hence, we expect the norm of the resolvent to be proportional to R^2.

The resolvent of (5.5) is given by

$$\mathscr{R}(s) = (s\boldsymbol{I} - \mathscr{L})^{-1} = \begin{pmatrix} \dfrac{R}{sR+1} & 0 \\ \dfrac{R^2}{(sR+1)(sR+2)} & \dfrac{R}{sR+2} \end{pmatrix}. \tag{5.6}$$

It follows that $\|\mathscr{R}(0)\| \sim R^2$, which is what we expected from $\|\mathrm{e}^{t\mathscr{L}}\|$.

For pipe Poiseuille flow, numerical computations by Schmid & Henningson (1994) indicate that a perturbation with $\alpha = 0$ and $n = 1$ gives the largest transient growth. For this perturbation, the numerical results of both Schmid & Henningson (1994) and Meseguer & Trefethen (2003) are

$$\sup_{t>0} \|\mathrm{e}^{t\mathscr{L}}\|_{L^2} \sim R,$$

with the maximum occurring at a time $t \sim R$. We may thus expect the L^2-norm of the resolvent (at least at $s = 0$) to be proportional to R^2, which is confirmed by the extensive numerical computations of Meseguer & Trefethen (2003). Thus, it is likely that for this perturbation, i.e. for $\alpha = 0$ and $n = 1$, our resolvent bound in theorem 4.2 is sharp.

For further results relating the transient growth to the ε-pseudospectrum and the resolvent, see Reddy, Schmid & Henningson (1993) and Trefethen *et al.* (1993).

6. Discussion

In §4, we derived bounds on the resolvent for certain combinations of the wavenumbers and the Reynolds number. We also showed how these results can be used to give a bound on the resolvent of the original problem by using (4.17). Here, we discuss what further results are required to obtain a rigorous bound on the resolvent in the entire unstable half-plane.

By theorem 3.1, we already have a bound on the resolvent when $s \in \Sigma$. In the remaining part of the unstable half-plane, Σ^- defined by (3.1*b*), we would like to use (4.17) to obtain a bound.

The resolvent of the Fourier transformed problem depends on four parameters, α, n, R and s. For convenience, we choose instead the parameters α, n, αR and sR. We would thus like to bound $\|\tilde{\mathscr{R}}(s, \alpha, n)\|$ in the parameter domain

$$\Upsilon = \{\alpha, \alpha R \in \mathbb{R},\ n \in \mathbb{Z},\ sR \in \mathbb{C}\}.$$

The bound should be valid at least for all $s \in \Sigma^-$, i.e. we may assume $\mathrm{Re}(s) \leqslant 7$, $|s| \leqslant CR$ etc. if needed.

Deriving an analytical bound on the resolvent in the entire parameter domain, Υ, would probably be extremely complicated. Instead, assume that for $s \in \Sigma^-$, we could prove that there is some large constant C such that $\|\tilde{\mathscr{R}}(s, \alpha, n)\|$ is bounded when

$$|sR| + |\alpha R| \geqslant C^2. \tag{6.1}$$

Using theorem 4.2, it would be sufficient if the proof were valid for $|\alpha|^3 \leqslant 4|\alpha R|$.

Since, by theorem 4.2, we already have a bound when $|\alpha R| \leqslant n^2/16$, this would imply a bound also when $|n| \geqslant 4C$ for all values of α, αR and sR. Also, since we assume $R \geqslant 1$, (6.1) holds when $|\alpha| \geqslant C^2$. Hence, in order to cover the entire parameter domain, Υ, it would be sufficient to obtain bounds in the parameter domain

$$\Upsilon^- = \{|\alpha| \in [0, C^2],\ |\alpha R| \in [1/16, C^2],\ |n| \in [0, 4C] \cap \mathbb{Z}, |sR| \in [0, C^2]\}.$$

This is a bounded parameter domain which opens for the possibility of using rigorous numerical computations to cover it. These computations would have to be combined with analytical results, since Υ^- still contains an infinite number of parameter values.

The analytical results should be such that if a numerical bound on the resolvent is valid at a point $(\alpha^*, \alpha R^*, n^*, sR^*) \in \Upsilon^-$, a bound follows in some neighbourhood of this point. That is, given the numerical bound, an analytical bound follows for all combinations of α, αR, n and sR such that $g(\alpha, \alpha R, n, sR) \leqslant \varepsilon$, where g is a continuous function with $g(\alpha^*, \alpha R^*, n^*, sR^*) = 0$. The value of ε could depend on the point chosen, but should be explicitly computable. Also, note that all computations would have to be done with rigorous numerical methods using interval arithmetic.

For plane Couette flow, a resolvent bound has been derived under a condition similar to (6.1) at the point $s = 0$ (Åsén & Kreiss 2005). The remaining parameter domain is bounded. Analytical results of the type described above were derived by Åsén (2005), making it possible to prove a rigorous bound on the resolvent at the point $s = 0$ in the unstable half-plane.

Remark. Numerical computations by Schmid & Henningson (1994) indicate that when $n \neq 0$, the transient growth decreases with increasing αR. Also, computations by Meseguer & Trefethen (2003) suggest that the resolvent is maximized at $s = 0$. This indicates that a resolvent bound could be derived analytically when (6.1) holds, if C is chosen large enough.

Remark 2. The results derived in this paper can easily be improved; Σ^- can be made smaller and theorem 4.2 can cover a larger parameter domain. In order to keep the technicalities at a minimum, we have not aimed at making the results as sharp as possible. However, if the desired analytical results discussed in this section are derived and rigorous numerical computations are to be used in a bounded parameter domain, making the results as sharp as possible could be important in order to reduce the amount of computation required.

7. Conclusions

In this paper, we derive bounds on the resolvent of pipe Poiseuille flow. In a large part of the unstable half-plane, a bound is obtained by using integration by parts, see theorem 3.1. However, the size of the remaining part increases with increasing Reynolds number. Also, the theorem does not cover the point $s = 0$, which is where numerical computations indicate that the resolvent is maximized (Meseguer & Trefethen 2003).

In order to obtain a bound on the resolvent in the remaining part of the unstable half-plane, we consider the linearized Navier–Stokes equations, Fourier transformed in the axial and azimuthal directions. We show, as was done by Liefvendahl & Kreiss (2003), that the norm of the resolvent of the original problem is obtained by maximizing the norm of the resolvent of the Fourier transformed problem with respect to the two wavenumbers, α and n.

We derive bounds on the norm of the resolvent for different combinations of the axial wavenumber, α, the azimuthal wavenumber, n, and the Reynolds number, R. The results are presented in theorem 4.2. In particular, the theorem is valid for perturbations with $\alpha=0$ and $n=1$, which from numerical computations is believed to yield the largest transient growth (Schmid & Henningson 1994) and the largest resolvent (Trefethen *et al.* 1999). Also, our resolvent bound, $\|\tilde{\mathscr{R}}(s,\alpha,n)\|\leqslant CR^2$, has the same dependence on the Reynolds number as the results from the numerical computations.

The conditions (4.14*a–c*) of theorem 4.2 include perturbations of different physical properties. For instance, structures with weak streamwise dependency are covered by (4.14*a*) or (4.14*c*). The velocity field of the perturbation that gives the largest transient growth, determined by Schmid & Henningson (1994), is of this type. It consists of two counter-rotating vortices near the centre of the pipe. Further, (4.14*b*) is valid for perturbations with large axial wavenumber compared to the Reynolds number, i.e. for perturbations which are severely affected by viscosity.

The remaining parameter domain for α, n, R and s is still unbounded. We briefly discuss what further results are required in order to obtain a bounded parameter domain. Under the conditions for deriving such a result, we also discuss how rigorous numerical computations could be used to obtain a bound on the resolvent in the remaining bounded parameter domain. This would result in a bound on the resolvent in the entire unstable half-plane, which would also serve as the first proof of linear stability of pipe Poiseuille flow.

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Appendix. Proof of lemma 4.3

We will use (4.10*a,b*) and integration by parts to prove lemma 4.3. More precisely, we will show that there is a constant C independent of α, n and R, such that

$$\|r^{-1}\Phi\|^2+\|k^{-1}r^{-1}\Phi'\|^2+\|kr\Omega\|^2\leqslant\frac{CR^4}{n^2}(\|r^{-1}\xi\|^2+\|k^{-1}r^{-1}\xi'\|^2+\|kr\chi\|^2) \quad (\mathrm{A}\,1)$$

holds when $n^2\geqslant16|\alpha R|$. Here, $\|\cdot\|$ is the norm induced by the scalar product (4.5). The bound $\|\tilde{\mathscr{R}}(s,\alpha,n)\|\leqslant CR^2$ then follows from (4.13). Also, this proves that $\|\tilde{\mathscr{R}}(s,\alpha,n)\|\rightarrow0$ as $|n|\rightarrow\infty$ and the lemma is proved.

When $n=0$, the lemma is valid only for $\alpha=0$. In this case, lemma 4.2 gives the desired resolvent bound. Thus, we assume $|n|\geqslant1$ in the remainder of the proof.

We use a prime to denote differentiation with respect to r. Although we use only integration by parts, the r appearing in the scalar product (4.5) makes the proof somewhat technical. Also, since r appears in the denominator at several places in the equations, the boundary terms appearing from using integration by parts must be handled with care.

First, we multiply (4.10a) with $k^{-2}r^{-2}R$, scalar multiply with Φ and take the real part. Note that

$$\left\langle \Phi, \frac{k^2 \mathrm{i}\alpha r R}{k^2 r^2} \left(\frac{U'}{k^2 r}\right)' \Phi \right\rangle \in \mathrm{Im}$$

and, using integration by parts and the boundary conditions (4.12),

$$\mathrm{Re}\left(-\left\langle \Phi, \frac{sR}{k^2 r^2} \mathrm{T}\Phi \right\rangle\right) = \mathrm{Re}(sR)(\|r^{-1}\Phi\|^2 + \|k^{-1}r^{-1}\Phi'\|^2).$$

Since $\mathrm{Re}(Rs) \geqslant 0$, we thus have

$$\mathrm{Re}\left(\left\langle \Phi, \frac{1}{k^2 r^2} \mathrm{T}^2 \Phi \right\rangle\right) \leqslant \left|\mathrm{Re}\left(\left\langle \Phi, \frac{\mathrm{i}\alpha U R}{k^2 r^2} \mathrm{T}\Phi \right\rangle\right)\right| + \left|\mathrm{Re}\left(\left\langle \Phi, \frac{2\alpha n}{k^2 r^2} \mathrm{T}\Omega \right\rangle\right)\right| + \left|\mathrm{Re}\left(\left\langle \Phi, \frac{R}{k^2 r^2} \mathrm{T}\xi \right\rangle\right)\right|. \quad (\mathrm{A}\,2)$$

Using integration by parts, we will derive a lower bound on the term on the left-hand side of (A 2) and upper bounds on the terms on the right-hand side of (A 2). Using the definition of T (4.11a), we have

$$\left\langle \Phi, \frac{1}{k^2 r^2} \mathrm{T}^2 \Phi \right\rangle = \int_0^1 \overline{\Phi} \left(\frac{1}{k^2 r} (\mathrm{T}\Phi)'\right)' \mathrm{d}r - \left\langle \Phi, \frac{1}{r^2} \mathrm{T}\Phi \right\rangle. \quad (\mathrm{A}\,3)$$

The integral is rewritten, using integration by parts, as

$$\int_0^1 \overline{\Phi} \left(\frac{1}{k^2 r} (\mathrm{T}\Phi)'\right)' \mathrm{d}r = \left[\overline{\Phi} \frac{1}{k^2 r} (\mathrm{T}\Phi)'\right]_{r=0}^{r=1} - \int_0^1 \overline{\Phi}' \frac{1}{k^2 r} (\mathrm{T}\Phi)' \,\mathrm{d}r$$
$$= \left[\overline{\Phi} \frac{1}{k^2 r} (\mathrm{T}\Phi)' - \overline{\Phi}' \frac{1}{k^2 r} \mathrm{T}\Phi\right]_{r=0}^{r=1} + \int_0^1 \left(\overline{\Phi}'' \frac{1}{k^2 r} + \overline{\Phi}' \left(\frac{1}{k^2 r}\right)'\right) \mathrm{T}\Phi \,\mathrm{d}r. \quad (\mathrm{A}\,4)$$

From the definition of T, (4.11a), we have

$$\frac{1}{kr} \mathrm{T}\Phi = \frac{1}{kr} \Phi'' + k \left(\frac{1}{k^2 r}\right)' \Phi' - \frac{k}{r} \Phi.$$

Using this, the integral on the right-hand side of (A 4) can be written as

$$\int_0^1 \left(\overline{\Phi}'' \frac{1}{k^2 r} + \overline{\Phi}' \left(\frac{1}{k^2 r}\right)'\right) \mathrm{T}\Phi \,\mathrm{d}r = \int_0^1 \left(\frac{1}{kr} \mathrm{T}\overline{\Phi} + \frac{k}{r} \Phi\right) \frac{1}{kr} \mathrm{T}\Phi r \,\mathrm{d}r$$
$$= \|k^{-1}r^{-1}\mathrm{T}\Phi\|^2 + \left\langle \Phi, \frac{1}{r^2} \mathrm{T}\Phi \right\rangle. \quad (\mathrm{A}\,5)$$

It follows from (A 3), (A 4) and (A 5) that

$$\left\langle \Phi, \frac{1}{k^2 r^2} \mathrm{T}^2 \Phi \right\rangle = \|k^{-1}r^{-1}\mathrm{T}\Phi\|^2 + \left[\overline{\Phi} \frac{1}{k^2 r} (\mathrm{T}\Phi)' - \overline{\Phi}' \frac{1}{k^2 r} \mathrm{T}\Phi\right]_{r=0}^{r=1}. \quad (\mathrm{A}\,6)$$

We have kept the boundary terms which appear from using integration by parts, since it is not obvious that they are zero or even that they are bounded. We will now show that the boundary terms are zero.

Expanding the boundary terms in (A 6) gives

$$\overline{\Phi}\frac{1}{k^2r}(\mathrm{T}\Phi)' - \overline{\Phi}'\frac{1}{k^2r}\mathrm{T}\Phi = \frac{2n^2\overline{\Phi}\Phi}{k^2r^4} - \frac{\overline{\Phi}\Phi'}{r} + \frac{4n^4\overline{\Phi}\Phi'}{k^6r^7} - \frac{6n^2\overline{\Phi}\Phi'}{k^4r^5} + \frac{\overline{\Phi}\Phi'}{k^2r^3}$$
$$+ \frac{2n^2\overline{\Phi}\Phi''}{k^4r^4} - \frac{\overline{\Phi}\Phi''}{k^2r^2} + \frac{\overline{\Phi}\Phi'''}{k^2r} + \frac{\overline{\Phi}'\Phi}{r} - \frac{2n^2\overline{\Phi}'\Phi'}{k^4r^4} + \frac{\overline{\Phi}'\Phi'}{k^2r^2} - \frac{\overline{\Phi}'\Phi''}{k^2r}.$$

At $r=1$ we have the boundary condition, (4.12), $\Phi=\Phi'=0$ and all terms above are thus zero at $r=1$. At $r=0$, we have for $|n|=1$ that $\Phi=0$ and $\Phi'<\infty$ and for $|n|\geqslant 2$ that $\Phi=\Phi'=0$. Also, since $n\neq 0$, $k^{-a}r^{-b}|_{r=0}$ is bounded if $a=b$ and zero if $a>b$. Hence, the remaining terms as $r\to 0$ are

$$\left.\overline{\Phi}\frac{1}{k^2r}(\mathrm{T}\Phi)' - \overline{\Phi}'\frac{1}{k^2r}\mathrm{T}\Phi\right|_{r\to 0} = \frac{2n^2\overline{\Phi}\Phi}{k^2r^4} - \frac{\overline{\Phi}\Phi'}{r} + \frac{4n^4\overline{\Phi}\Phi'}{k^6r^7} - \frac{6n^2\overline{\Phi}\Phi'}{k^4r^5} + \frac{\overline{\Phi}\Phi'}{k^2r^3}$$
$$\left. + \frac{\overline{\Phi}'\Phi}{r} - \frac{2n^2\overline{\Phi}'\Phi'}{k^4r^4} + \frac{\overline{\Phi}'\Phi'}{k^2r^2}\right|_{r\to 0}. \quad (\mathrm{A}\,7)$$

At $r=0$ we have by l'Hospital's rule and the boundary conditions that

$$\left.\frac{\overline{\Phi}'\Phi}{r} - \frac{\overline{\Phi}\Phi'}{r}\right|_{r\to 0} = \left.\frac{\overline{\Phi}''\Phi + \overline{\Phi}'\Phi' - \overline{\Phi}'\Phi' - \overline{\Phi}\Phi''}{1}\right|_{r\to 0} = 0. \quad (\mathrm{A}\,8)$$

Again, using l'Hospital's rule, $k' = -n^2k^{-1}r^{-3}$ and $k^ar^a|_{r=0}=n^a$, we have (below, we assume $r\to 0$ in all expressions)

$$\frac{2n^2\overline{\Phi}\Phi}{k^2r^4} = 2n^2\frac{\overline{\Phi}'\Phi + \overline{\Phi}\Phi'}{4k^2r^3 - 2n^2r} = 2n^2\frac{\overline{\Phi}''\Phi + 2\overline{\Phi}'\Phi' + \overline{\Phi}\Phi''}{4(3k^2r^2 - 2n^2) - 2n^2} = 2\overline{\Phi}'\Phi'\Big|_{r=0}, \quad (\mathrm{A}\,9a)$$

$$\frac{4n^4\overline{\Phi}\Phi'}{k^6r^7} = 4n^4\frac{\overline{\Phi}'\Phi' + \overline{\Phi}\Phi''}{7k^6r^6 - 6n^2k^4r^4} = 4\left.\frac{\overline{\Phi}'\Phi'}{n^2}\right|_{r=0}, \quad (\mathrm{A}\,9b)$$

$$-\frac{6n^2\overline{\Phi}\Phi'}{k^4r^5} = -6n^2\frac{\overline{\Phi}'\Phi' + \overline{\Phi}\Phi''}{5k^4r^4 - 4n^2k^2r^2} = -6\left.\frac{\overline{\Phi}'\Phi'}{n^2}\right|_{r=0}, \quad (\mathrm{A}\,9c)$$

$$\frac{\overline{\Phi}\Phi'}{k^2r^3} = \frac{\overline{\Phi}'\Phi' + \overline{\Phi}\Phi''}{3k^2r^2 - 2n^2} = \left.\frac{\overline{\Phi}'\Phi'}{n^2}\right|_{r=0}, \quad (\mathrm{A}\,9d)$$

$$-\frac{2n^2\overline{\Phi}'\Phi'}{k^4r^4} = -2\left.\frac{\overline{\Phi}'\Phi'}{n^2}\right|_{r=0}, \quad (\mathrm{A}\,9e)$$

$$\frac{\overline{\Phi}'\Phi'}{r^2k^2} = \left.\frac{\overline{\Phi}'\Phi'}{n^2}\right|_{r=0}. \quad (\mathrm{A}\,9f)$$

Hence, from (A 7), (A 8) and (A 9*a*–*f*), it follows that

$$\left.\overline{\Phi}\frac{1}{k^2r}(\mathrm{T}\Phi)' - \overline{\Phi}'\frac{1}{k^2r}\mathrm{T}\Phi\right|_{r\to 0} = 2\left(1 - \frac{1}{n^2}\right)\overline{\Phi}'\Phi'|_{r=0} = 0,$$

where in the last step we use $1-n^{-2}=0$ if $|n|=1$ and $\Phi'|_{r=0}=0$ if $|n|\geqslant 2$. We have thus shown that the boundary terms in (A 6) are zero, and it follows that

$$\left\langle \Phi, \frac{1}{k^2r^2}\mathrm{T}^2\Phi \right\rangle = \|k^{-1}r^{-1}\mathrm{T}\Phi\|^2. \quad (\mathrm{A}\,10)$$

We will now use integration by parts to derive an expression involving the desired terms $\|r^{-1}\Phi\|$ and $\|k^{-1}r^{-1}\Phi'\|$ which appear on the left-hand side of (A 1). First, note that since $\max_{r\in[0,1]} n^2/k^2 \leqslant 1$, we have

$$n^2\|k^{-2}r^{-1}\mathrm{T}\Phi\|^2 \leqslant \|k^{-1}r^{-1}\mathrm{T}\Phi\|^2. \tag{A 11}$$

From the definition of T (4.11a), we have

$$\begin{aligned}\|k^{-2}r^{-1}\mathrm{T}\Phi\|^2 &= \int_0^1 \left(\left(\frac{1}{k^2r}\overline{\Phi}'\right)' - \frac{1}{r}\overline{\Phi}\right)\left(\left(\frac{1}{k^2r}\Phi'\right)' - \frac{1}{r}\Phi\right) r\,\mathrm{d}r \\ &= \|r^{-1}\Phi\|^2 + \|(k^{-2}r^{-1}\Phi')'\|^2 \\ &\quad - \int_0^1 \overline{\Phi}\left(\frac{1}{k^2r}\Phi'\right)'\,\mathrm{d}r - \int_0^1 \Phi\left(\frac{1}{k^2r}\overline{\Phi}'\right)'\,\mathrm{d}r. \end{aligned} \tag{A 12}$$

Using integration by parts and the boundary conditions gives

$$-\int_0^1 \overline{\Phi}\left(\frac{1}{k^2r}\Phi'\right)'\,\mathrm{d}r = -\left[\overline{\Phi}\frac{1}{k^2r}\Phi'\right]_{r=0}^{r=1} + \int_0^1 \overline{\Phi}'\frac{1}{k^2r^2}\Phi' r\,\mathrm{d}r = \|k^{-1}r^{-1}\Phi'\|^2. \tag{A 13}$$

Thus, using (A 10), (A 11), (A 12) and (A 13), we have the following lower bound on the term on the left-hand side of (A 2).

$$\mathrm{Re}\left\langle \Phi, \frac{1}{k^2r^2}\mathrm{T}^2\Phi\right\rangle \geqslant n^2\left(\|r^{-1}\Phi\|^2 + 2\|k^{-1}r^{-1}\Phi'\|^2 + \|(k^{-2}r^{-1}\Phi')'\|^2\right). \tag{A 14}$$

For the first term on the right-hand side of (A 2), integration by parts yields

$$\begin{aligned}\left\langle \Phi, \frac{\mathrm{i}\alpha U}{k^2r^2}\mathrm{T}\Phi\right\rangle &= \int_0^1 \overline{\Phi}\mathrm{i}\alpha(1-r^2)\left(\frac{1}{r}\left(\frac{1}{k^2r}\Phi'\right)' - \frac{1}{r^2}\Phi\right) r\,\mathrm{d}r \\ &= \mathrm{i}\alpha(\|\Phi\|^2 - \|r^{-1}\Phi\|^2) + \left[\overline{\Phi}\mathrm{i}\alpha(1-r^2)\frac{1}{k^2r}\Phi'\right]_{r=0}^{r=1} \\ &\quad -\mathrm{i}\alpha\int_0^1 \left(\overline{\Phi}'(1-r^2) - \overline{\Phi}2r\right)\frac{1}{k^2r}\Phi'\,\mathrm{d}r \\ &= \mathrm{i}\alpha(\|\Phi\|^2 - \|r^{-1}\Phi\|^2 + \|k^{-1}\Phi'\|^2 - \|k^{-1}r^{-1}\Phi'\|^2) \\ &\quad + 2\mathrm{i}\alpha\int_0^1 \frac{\overline{\Phi}}{r}\frac{\Phi'}{kr}\frac{r}{k}r\,\mathrm{d}r. \end{aligned}$$

Since $\max_{r\in[0,1]} r/k \leqslant 1/|n|$, this gives

$$\left|\mathrm{Re}\left(\left\langle \Phi, \frac{\mathrm{i}\alpha UR}{k^2r^2}\mathrm{T}\Phi\right\rangle\right)\right| \leqslant \frac{2|\alpha R|}{|n|}\|r^{-1}\Phi\|\,\|k^{-1}r^{-1}\Phi'\|. \tag{A 15}$$

Similarly, using integration by parts, the following results for the other terms in (A 2) are easily obtained

$$\left|\mathrm{Re}\left\langle \Phi, \frac{2\alpha n}{k^2r^2}\mathrm{T}\Omega\right\rangle\right| \leqslant \frac{2|\alpha|}{|n|}(\|r^{-1}\Phi\|\,\|k^2r\Omega\| + \|k^{-1}r^{-1}\Phi'\|\,\|kr\Omega'\|), \tag{A 16a}$$

$$\left|\mathrm{Re}\left\langle \Phi, \frac{R}{k^2r^2}\mathrm{T}\xi\right\rangle\right| \leqslant R(\|r^{-1}\Phi\|\,\|r^{-1}\xi\| + \|k^{-1}r^{-1}\Phi'\|\,\|k^{-1}r^{-1}\xi'\|). \tag{A 16b}$$

Hence, using (A 14), (A 15) and (A 16*a,b*) in (A 2) yields

$$n^2(\|r^{-1}\Phi\|^2 + 2\|k^{-1}r^{-1}\Phi'\|^2) \leqslant \frac{2|\alpha R|}{|n|}\|r^{-1}\Phi\|\,\|k^{-1}r^{-1}\Phi'\| + \frac{2|\alpha|}{|n|}(\|r^{-1}\Phi\|\,\|k^2 r\Omega\| + \|k^{-1}r^{-1}\Phi'\|\,\|kr\Omega'\|) + R(\|r^{-1}\Phi\|\,\|r^{-1}\xi\| + \|k^{-1}r^{-1}\Phi'\|\,\|k^{-1}r^{-1}\xi'\|).$$

Using $ab \leqslant a^2/(2\varepsilon) + \varepsilon b^2/2$, valid for $\varepsilon > 0$, on the right-hand side gives

$$n^2(\|r^{-1}\Phi\|^2 + 2\|k^{-1}r^{-1}\Phi'\|^2) \leqslant \frac{2|\alpha R|}{|n|^3} n^2\left(\tfrac{1}{4}\|r^{-1}\Phi\|^2 + \|k^{-1}r^{-1}\Phi'\|^2\right) + \frac{n^2}{4}\|r^{-1}\Phi\|^2 + \frac{4\alpha^2}{n^4}\|k^2 r\Omega\|^2 + \frac{n^2}{2}\|k^{-1}r^{-1}\Phi'\|^2 + \frac{2\alpha^2}{n^4}\|kr\Omega'\|^2 + \frac{n^2}{8}(\|r^{-1}\Phi\|^2 + \|k^{-1}r^{-1}\Phi'\|^2) + \frac{2R^2}{n^2}(\|r^{-1}\xi\|^2 + \|k^{-1}r^{-1}\xi'\|^2).$$

The condition $n^2 \geqslant 16|\alpha R|$ (and $|n| \geqslant 1$) is more than enough to ensure that

$$n^2(\|r^{-1}\Phi\|^2 + 2\|k^{-1}r^{-1}\Phi'\|^2) \leqslant \frac{8\alpha^2}{n^4}\|k^2 r\Omega\|^2 + \frac{4\alpha^2}{n^4}\|kr\Omega'\|^2 + \frac{4R^2}{n^2}(\|r^{-1}\xi\|^2 + \|k^{-1}r^{-1}\xi'\|^2). \quad \text{(A 17)}$$

We will now derive a similar result for Ω by using (4.10*b*). We multiply (4.10*b*) by $(-k^2r^2R)$, scalar multiply by Ω and take the real part. Since

$$\langle \Omega, \mathrm{i}\alpha U R k^2 r^2 \Omega\rangle \in \mathrm{Im},$$
$$\mathrm{Re}(\langle \Omega, sRk^2r^2\Omega\rangle) = \mathrm{Re}(sR)\|kr\Omega\|^2,$$

and $\mathrm{Re}(Rs) \geqslant 0$, this yields

$$-\mathrm{Re}\left(\left\langle \Omega, k^2r^2\mathrm{S}\Omega\right\rangle\right) \leqslant \left|\mathrm{Re}\left(\left\langle \Omega, \frac{2\alpha n}{k^2r^2}\mathrm{T}\Phi\right\rangle\right)\right| + \left|\mathrm{Re}\left(\left\langle \Omega, \frac{R\mathrm{i}nU'}{r}\Phi\right\rangle\right)\right| + \left|\mathrm{Re}(\langle \Omega, k^2r^2R\chi\rangle)\right|. \quad \text{(A 18)}$$

For the term on the left-hand side of (A 18), using integration by parts and the definition of S (4.11*b*) yields

$$-\langle \Omega, k^2r^2\mathrm{S}\Omega\rangle = -\int_0^1 \overline{\Omega}\left(\frac{1}{r}(k^2r^3\Omega')' - k^4r^2\Omega\right) r\,\mathrm{d}r = \|k^2r\Omega\|^2 - \left[\overline{\Omega}k^2r^3\Omega'\right]_{r=0}^{r=1} + \int_0^1 \overline{\Omega}'k^2r^2\Omega' r\,\mathrm{d}r = \|k^2r\Omega\|^2 + \|kr\Omega'\|^2. \quad \text{(A 19)}$$

The terms on the right-hand side of (A 18) are easily bounded from above by

$$\left|\mathrm{Re}\left(\left\langle \Omega, \frac{2\alpha n}{k^2r^2}\mathrm{T}\Phi\right\rangle\right)\right| \leqslant \frac{2|\alpha|}{|n|}(\|kr\Omega'\|\,\|k^{-1}r^{-1}\Phi'\| + \|k^2r\Omega\|\,\|r^{-1}\Phi\|), \quad \text{(A 20}a\text{)}$$

$$\left|\mathrm{Re}\left(\left\langle \Omega, \frac{R\mathrm{i}nU'}{r}\Phi\right\rangle\right)\right| \leqslant 2R\|kr\Omega\|\,\|r^{-1}\Phi\|, \quad \text{(A 20}b\text{)}$$

$$\left|\mathrm{Re}\left(\left\langle \Omega, k^2r^2R\chi \right\rangle\right)\right| \leqslant R\|kr\Omega\|\,\|kr\chi\|. \tag{A 20c}$$

Using (A 19) and (A 20*a–c*) in (A 18) thus gives

$$\|k^2r\Omega\|^2+\|kr\Omega'\|^2 \leqslant \frac{2|\alpha|}{|n|}(\|kr\Omega'\|\,\|k^{-1}r^{-1}\Phi'\|+\|k^2r\Omega\|\,\|r^{-1}\Phi\|) + 2R\|kr\Omega\|\,\|r^{-1}\Phi\|+R\|kr\Omega\|\,\|kr\chi\|.$$

As before, using $ab\leqslant a^2/(2\varepsilon)+\varepsilon b^2/2$ on the right-hand side gives

$$\|k^2r\Omega\|^2+\|kr\Omega'\|^2 \leqslant \tfrac{1}{4}\|kr\Omega'\|^2+\frac{4\alpha^2}{n^2}\|k^{-1}r^{-1}\Phi'\|^2+\tfrac{1}{8}\|k^2r\Omega\|^2+\frac{8\alpha^2}{n^2}\|r^{-1}\Phi\|^2 + \tfrac{1}{2}\|kr\Omega\|^2+2R^2\|r^{-1}\Phi\|^2+\tfrac{1}{8}\|kr\Omega\|^2+2R^2\|kr\chi\|^2.$$

Collecting terms and using $\|kr\Omega\|\leqslant n^2\|kr\Omega\|\leqslant\|k^2r\Omega\|$ yields

$$\|k^2r\Omega\|^2+3\|kr\Omega'\|^2 \leqslant \frac{16\alpha^2}{n^2}\left(\|k^{-1}r^{-1}\Phi'\|^2+2\|r^{-1}\Phi\|^2\right) + 8R^2\|r^{-1}\Phi\|^2+8R^2\|kr\chi\|^2. \tag{A 21}$$

From (A 17), we have

$$8R^2\|r^{-1}\Phi\|^2 \leqslant \frac{64\alpha^2R^2}{n^6}\|k^2r\Omega\|^2+\frac{32\alpha^2R^2}{n^6}\|kr\Omega'\|^2 + \frac{32R^4}{n^4}(\|r^{-1}\xi\|^2+\|k^{-1}r^{-1}\xi'\|^2). \tag{A 22}$$

Using (A 22) on the right-hand side of (A 21) and adding the result to (A 17) gives after rearranging the terms

$$n^2\left(1-\frac{32\alpha^2}{n^4}\right)\|r^{-1}\Phi\|^2+n^2\left(2-\frac{16\alpha^2}{n^4}\right)\|k^{-1}r^{-1}\Phi'\|^2 + \left(1-\frac{8\alpha^2}{n^4}-\frac{64\alpha^2R^2}{n^6}\right)\|k^2r\Omega\|^2+\left(3-\frac{4\alpha^2}{n^4}-\frac{32\alpha^2R^2}{n^6}\right)\|kr\Omega'\|^2 \leqslant C(R^4(\|r^{-1}\xi\|^2+\|k^{-1}r^{-1}\xi'\|^2)+R^2\|kr\chi\|^2).$$

The condition $n^2\geqslant 16|\alpha R|$ (and $R\geqslant 1$, i.e. $n^2\geqslant 16|\alpha|$) is enough to ensure that the term in parentheses on the left-hand side is positive. Using also $n^2\|kr\Omega\|^2\leqslant\|k^2r\Omega\|^2$, (A 1) follows and the lemma is proved.

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